Advanced Math. Models & Applications, V.1, N.1, 2016, pp.1-13

SOME INEQUALITIES FOR ALGEBRA OF FRACTIONS AND ITS APPLICATIONS

Necla Kircali Gursoy[1], Urfat Nuriyev[2]

[1]Tire-Kutsan Vocational School, Ege University, 35900, Tire, Izmir, Turkey
[2]Faculty of Science, Department of Mathematics, Ege University, 35100, Izmir, Turkey
e-mail: kircalinecla@gmail.com, necla.kircali.gursoy@ege.edu.tr

Abstract. In this study, we prove some inequalities on algebra of fractions constructed by coordinatewise operations defined on fractions. Inverse property of coordinatewise sum operation is found out and inversion intervals are determined. Based on this property, two heuristic strategies are suggested for solving unconstrained linear fractional boolean programming problem. Different inequalities are proved for determining lower and upper bounds of solutions which are found through these strategies.

Keywords: Algebra of Fractions, Boolean Programming, Inequalities, Linear Fractional Programming.

AMS Subject Classification: 90C32, 90C09, 08A70, 26D15.

1. Introduction

Linear fractional programming (LFP) problems are a special type of non-linear programming problems. In LFP, the objective function is a ratio of linear functions and the constrains are linear functions. In such models, under linear restriction, one of the extremal values of two linear form (function's) ratio is found [3]. Many decision problems in economical and technical systems can be shown as models of Linear Fractional Programming [3, 7]. In real life situations, linear fractional models are applied more in decision making such as construction planning, economic and commercial planning, health care and hospital planning. An integer programming where all the variables are restricted to a value of 0 or 1 is called 0-1 integer programming or boolean programming. A significant part of the LFP problems consists of Boolean Linear Fractional Programming problems (BLFP). The models of BLFP are used in the process of the construction of technical and economical systems, in planning of huge energy and production complexes, and in modelling computer supported management systems [2, 3, 7, 10, 17]. Hence, improving new and fast solution methods for LFP problems, and designing effective algorithms are of great importance.

In the literature, several approaches and strategy are suggested to solve LFP and BLFP problems. Bajalinov have recommended many methods to solve LFP problems [3]. Isbell and Marlow first identified an example of LFP problem and solved it by a sequence of linear programming problems [13]. Charnes and Cooper [6] considered variable transformation method to solve LFP and the updated objective function method were developed for solving the LFP problem by Bitran and Novaes [5]. Gilmore and Gomory [8], Martos [15], Swarup [22], Wagner and Yuan[26], Pandey and Punnen[19] and Sharma et al. [21] solved the LFP problem by various types of solution procedures based on the simplex method developed by Dantzig [9]. Tantawy proposed two different approaches namely; a feasible direction approach and a duality approach to solve the LFP problem [23, 24]. Mojtaba Borza et al.[16] solved the LFP problem with interval coefficients in objective function which is based on Charnes and Cooper technique [6]. Odior solved the LFP problem by algebraic approach which depends on the duality concept and the partial fractions [18]. Jayalakshmi and Pandian propose a new method namely, denominator objective restriction method for finding an optimal solution to LFP problems [14].

Addivite algorithm was developed for BLFP problems [4]. Puri and Swarp suggested the extreme point mathematical programming technique for solving BLFP as well as Integer Linear Fractional Programming (ILFP) [20]. Arefin's purpose is to solve such types of problem using an enumerative algorithm. He has used additive algorithm of Balas [4] for solving a class of BLFP problems [1]. Recently, Tantawy have introduced a new procedure for solving the ILP problem based on conjugate gradiant projection method with the help of the spirit of Gomory cut [25].

As a result, several methods have been developed for Boolean Linear Fractional Programming problems. In order to solve this type of problems, one of the operations (coordinatewise operations) on fractions performed by the "numerator - numerator", "denominator - denominator" principle is required. Taking this into consideration [11], operations done by this principle and denoted by the symbols $\oplus$, $\otimes$, $\odot$ are handled, their properties are investigated, and geometric representation are given. In this paper, some inequalities for operations defined above are proved. These inequalities lay a mathematical foundation to solve LFP and BLFP problems, and calculate the guarantee value of the algorithms constructed for these problems. We developed a greedy algorithm for 0-1 minimization Knapsack Problem and calculated guarantee value of this algorithm by using those inequalities [12].

1. Algebra of Fractions

The Fractions algebra defined in [11] are based on coordinatewise operations.

Definition 1. The operations denoted by $\oplus, \otimes$ and $\odot$ are defined on set

$$\Theta = \left\{ f \,\middle|\, f = \frac{a}{b}; a, b \in \mathbb{R} \right\}$$

as follows, where $\mathbb{R} = (-\infty, +\infty)$, $\oplus$ and $\otimes$ are binary, and $\odot$ is a unary operation.

$$f_1 \oplus f_2 = \frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2}$$

$$f_1 \otimes f_2 = \frac{a_1}{b_1} \otimes \frac{a_2}{b_2} = \frac{a_1 a_2}{b_1 b_2}$$

$$\lambda \odot f_2 = \lambda \odot \frac{a_1}{b_1} = \frac{\lambda a_1}{\lambda b_1}$$

for all $\lambda \in \mathbb{R}$, $f_1 = \frac{a_1}{b_1}$, $f_2 = \frac{a_2}{b_2} \in \Theta$.

Remark 1. We write $f \ominus g = f \oplus (-g)$.

Definition 2. The value of every element of Θ is called the module of the element; the module of f is denoted by $|f|$. Given a natural number k, the numerals $\frac{kp}{kq}$ and $\frac{p}{q}$ represent the same number, that is, their module is the same. For example, the numerals $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots$ represent the same value, namely $\frac{1}{3}$. Thus, $\left|\frac{1}{3}\right| = \left|\frac{2}{6}\right| = \left|\frac{3}{9}\right| = \left|\frac{4}{12}\right| = \ldots = 0.3333\ldots$

In other words, each element $f = \frac{a}{b}$ of Θ corresponds to a vector $\vec{t} = \{b, a\}$ in the plane $\mathbb{R} \times \mathbb{R}$. Then module of f is equal to the tangent of the angle that vector t makes on positive direction with Ox-axis, i.e., $|f| = tan\alpha$.

Remark 2. Let $f = \frac{m}{n} \in \Theta$. Then, the module of f is defined as

$$|f| = \begin{cases} +\infty, & m > 0, n = 0 \\ -\infty, & m < 0, n = 0 \end{cases}$$

Moreover, $m = n = 0$ if the module of f is undefined.

Theorem 1. [11] The system $(\Theta, \oplus, \odot, \otimes)$ is an algebra on the set of real numbers.

2. Some Basic Inequalities

The following inequalities to be proved will be of main importance for the next sections.

Proposition 1. For all $f_1, f_2 \in \Theta$, if $|f_1| \geq |f_2|$, then $|f_1| + |f_2| \geq |f_1| \geq |f_1 \oplus f_2| \geq |f_2|$.

Proof. Let $|f_1| = k_1$ and $|f_2| = k_2$, for $f_1 = \frac{a_1}{b_1}$, $f_2 = \frac{a_2}{b_2}$. Then we can write

$$|f_1| = \left|\frac{a_1}{b_1}\right| = k_1 \Rightarrow a_1 = b_1 k_1$$

$$|f_2| = \left|\frac{a_2}{b_2}\right| = k_2 \Rightarrow a_2 = b_2 k_2$$

$|f_1| \geq |f_2| \Rightarrow k_1 \geq k_2$.

Now from

$$|f_1 \oplus f_2| = \left|\frac{a_1}{b_1} \oplus \frac{a_2}{b_2}\right| = \left|\frac{a_1 + a_2}{b_1 + b_2}\right| = \left|\frac{b_1 k_1 + b_2 k_2}{b_1 + b_2}\right|$$
$$\geq \left|\frac{b_1 k_2 + b_2 k_2}{b_1 + b_2}\right| = \left|\frac{(b_1 + b_2) k_2}{b_1 + b_2}\right| = k_2 = |f_2|$$

We get

$$|f_1 \oplus f_2| \geq |f_2| \tag{1}$$

Also, from

$$|f_1 \oplus f_2| = \left|\frac{a_1}{b_1} \oplus \frac{a_2}{b_2}\right| = \left|\frac{a_1 + a_2}{b_1 + b_2}\right| = \left|\frac{b_1 k_1 + b_2 k_2}{b_1 + b_2}\right|$$
$$\leq \left|\frac{b_1 k_2 + b_2 k_1}{b_1 + b_2}\right| = \left|\frac{(b_1 + b_2) k_1}{b_1 + b_2}\right| = k_1 = |f_1|$$

We obtain

$$|f_1 \oplus f_2| \leq |f_1| \tag{2}$$

Thus, the required inequalities follow from (1) and (2).

Proposition 2. $|f_1| \geq |(\lambda \odot f_1) \oplus (\mu \odot f_2)| \geq |f_2|$ holds for $|f_1| \geq |f_2|$ and $\lambda, \mu \in \mathbb{R} \setminus \{0\}$.

Proof. By proposition 1, we have

$$|(\lambda \odot f_1) \oplus (\mu \odot f_2)| \geq |\mu \odot f_2| = |f_2| \tag{3}$$

$$|f_1| = |\lambda \odot f_1| \geq |(\lambda \odot f_1) \oplus (\mu \odot f_2)|. \tag{4}$$

Hence, the desired inequality follows from (3) and (4).

Proposition 3. Let $|f_1| \geq |f_2|$ and $\lambda_1 \mu_2 \geq \lambda_2 \mu_1$ for $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^+$ and $f_1, f_2 \in \mathbb{Q}_{\mathbb{R}^+}$. Then

$$|(\lambda_1 \odot f_1) \oplus (\mu_1 \odot f_2)| \geq |(\lambda_2 \odot f_1) \oplus (\mu_2 \odot f_2)|$$

holds.

Proof. We have

$$|(\lambda_1 \odot f_1) \oplus (\mu_1 \odot f_2)| - |(\lambda_2 \odot f_1) \oplus (\mu_2 \odot f_2)| = \frac{\lambda_1 k_1 b_1 + \mu_1 k_2 b_2}{(\lambda_1 b_1 + \mu_1 b_2)} - \frac{\lambda_2 k_1 b_1 + \mu_2 k_2 b_2}{\lambda_2 b_1 + \mu_2 b_2}$$
$$= \frac{(\lambda_1 \mu_2 - \lambda_2 \mu_1) b_1 b_2 (k_1 - k_2)}{(\lambda_1 b_1 + \mu_1 b_2)(\lambda_2 b_1 + \mu_2 b_2)}$$

for $|f_1| = \left|\frac{a_1}{b_1}\right| = k_1 \Rightarrow a_1 = k_1 b_1$ and $|f_2| = \left|\frac{a_2}{b_2}\right| = k_2 \Rightarrow a_2 = k_2 b_2$. Since $\lambda_1 \mu_2 \geq \lambda_2 \mu_1$, $\lambda_1 \mu_2 - \lambda_2 \mu_1 \geq 0$ and from $k_1 \geq k_2$, $k_1 - k_2 \geq 0$. Hence, we get

$$|(\lambda_1 \odot f_1) \oplus (\mu_1 \odot f_2)| - |(\lambda_2 \odot f_1) \oplus (\mu_2 \odot f_2)| \geq 0$$

Proposition 4. Let $|f| > |g|$ and $|f \oplus g| \leq |\varphi|$ for $f, g, \varphi \in \Theta$. Then either $|f| < |\varphi|$ or $|g| < |\varphi|$ if $|f| > |\varphi|$ holds.

Proof. If we set $f_1 = f$ and $f_2 = g$ then by proposition 1, we get $|f| \geq |f \oplus g| \geq |g|$. Under the hypothesis $|f \oplus g| \leq |\varphi|$, we have two cases to consider. In the first case, we have $|f| \geq |\varphi| \geq |f \oplus g| \geq |g|$. Then we have $|f| \geq |\varphi|$ while $|g| \leq |\varphi|$. In the second case, we have $|\varphi| \geq |f| \geq |f \oplus g| \geq |g|$. Then we get $|f| \leq |\varphi|$ then $|g| \leq |\varphi|$.

3. Inversion Properties

In this section, we examine whether the elements of Θ defined on $\mathbb{R}^+$ preserve the ordering under the operation $\oplus$. Here, we set

$$F_1 = F \oplus f_1,\ F_2 = F \oplus f_2$$
$$F^* = \max\{F_1, F_2\}$$
$$f^* = \min\{f_1, f_2\} = f_2$$

where $F, f_1, f_2 \in \Theta_{\mathbb{R}^+}$ and $|F| \geq |f_1| \geq |f_2|$.

Theorem 2. If $|f_1 \ominus f_2| \notin (|f_2|, |F_2|)$ then $|F_1| \geq |F_2|$.

Proof. Let $F = \dfrac{A}{B},\ f_1 = \dfrac{a+\Delta a}{b+\Delta b},\ f_2 = \dfrac{a}{b}$. Then we have

$$F_1 = F \oplus f_1 = \frac{A}{B} \oplus \frac{a+\Delta a}{b+\Delta b} = \frac{A+a+\Delta a}{B+b+\Delta b}$$
$$F_2 = F \oplus f_2 = \frac{A}{B} \oplus \frac{a}{b} = \frac{A+a}{B+b}$$
$$f_1 \ominus f_2 = \frac{\Delta a}{\Delta b}$$

We denote $F_1 - F_2$ and $f_1 - f_2$ by ΔF and Δf, respectively. Then,

$$\Delta F = F_1 - F_2 = \frac{A+a+\Delta a}{B+b+\Delta b} - \frac{A+a}{B+b} \tag{5}$$

$$\Delta f = f_1 - f_2 = \frac{a+\Delta a}{b+\Delta b} - \frac{a}{b} \tag{6}$$

In this case, to have

$$\Delta F = \rho.\Delta f, \qquad \rho > 0 \tag{7}$$

It suffices to show

$$\frac{\Delta a}{\Delta b} \notin \left[\frac{a}{b}, \frac{A+a}{B+b}\right] \tag{8}$$

to prove the theorem. Rearranging the equations (5) and (6), we get

$$\Delta F = F_1 - F_2 = \frac{A+a+\Delta a}{B+b+\Delta b} - \frac{A+a}{B+b}$$
$$= \frac{AB + Ab + aB + ab + B\Delta a + b\Delta a - AB - Ab - A\Delta b - aB - ab - a\Delta b}{(B+b+\Delta b)(B+b)}$$

$$=\frac{B\Delta a+b\Delta a-A\Delta b-a\Delta b}{(B+b+\Delta b)(B+b)}=\frac{\Delta a-\frac{(A+a)}{(B+b)}\Delta b}{(B+b+\Delta b)}$$
$$=\frac{\Delta a-F_2\Delta b}{B+b+\Delta b}$$

and

$$\Delta f = f_1 - f_2 = \frac{a+\Delta a}{b+\Delta b}-\frac{a}{b}=\frac{ab-b\Delta a-ab-a\Delta b}{b(\mathrm{b}+\Delta\,\mathrm{b})}$$
$$=\frac{\Delta a-\frac{a}{b}\Delta b}{b+\Delta b}=\frac{\Delta a-f_2\Delta b}{b+\Delta b}.$$

If we write $\Delta F=\frac{\Delta a-F_2\Delta b}{\Delta a-f_2\Delta b}.\frac{b+\Delta b}{B+b+\Delta b}.\frac{\Delta a-f_2\Delta b}{b+\Delta b}$, then we can express

$$\Delta F=\frac{\Delta a-F_2\Delta b}{\Delta a-f_2\Delta b}.\frac{b+\Delta b}{B+b+\Delta b}.\Delta f. \tag{9}$$

Then if we show that

$$\frac{\Delta a-F_2\Delta b}{\Delta a-f_2\Delta b}.\frac{b+\Delta b}{B+b+\Delta b}>0 \tag{10}$$

the inequality (9) becomes the same with (7). We have $b+\Delta b>0$ since the elements of denominator of linear fractional function come from $\mathbb{R}^+$. In this case in order for (10) to be satisfied, it will suffice that

$$\frac{\Delta a-F_2\Delta b}{\Delta a-f_2\Delta b}>0 \tag{11}$$

Now, (11) holds if and only if one the following conditions satisfied:

(i) $\frac{\Delta a}{\Delta b}>\max\{F_2, f_2\}=F_2=\frac{A+a}{B+b}$

(ii) $\frac{\Delta a}{\Delta b}<\min\{F_2, f_2\}=f_2=\frac{a}{b}$

But the conditions (i) and (ii) mean that (8) is satisfied.

Corollary 1. If $|f_1 \ominus f_2| \notin (|f_2|,|F_2|)$, then $|f_1 \ominus f_2| \notin (|f_1|,|F_1|)$.

Proof. By using Theorem 2, if $|f_1 \ominus f_2| \notin (|f_2|,|F_2|)$ then $|F_1|>|F_2|$. If $|f_1 \ominus f_2| \notin (|f_2|,|F_2|)$ then two cases are in question:

(i) If $f_\Delta \le |f_2|$, then $f_\Delta \le |f_1|$, since $|f_1| \ge |f_2|$.

(ii) If $f_\Delta \ge |F_2|$, then $f_\Delta \ge |F_1|$, since $F_1 = F_2 \oplus f_\Delta$.

Hence, we get $f_\Delta \notin (|f_1|,|F_1|)$.

Corollary 2. $|f_1 \ominus f_2| \notin (|f_1|,|F_1|)$ or $|f_1 \ominus f_2| \notin (|f_2|,|F_2|)$ then $|f_1 \ominus f_2| \in (|f^*|,|F^*|)$.

Proof. It follows from Theorem 2 and Corollary 1.

Definition 3. The interval corresponding to $M_{f_1, f_2} = \left(-\infty, \left| f^* \right|\right) \cup \left(\left|F^*\right|, +\infty\right)$ is called **Monotony Domain** and the interval remaining outside of it is called **Inversion Domain**, and is denoted by $U_{f_1, f_2} = \left(\left| f^* \right|, \left|F^*\right|\right)$.

As we see from this definition, if $|f_1| \ge |f_2|$ then $|F_1| \ge |F_2|$, where $F_1 = F \oplus f_1$ and $F_2 = F \oplus f_2$. In the Inversion domain we have exactly the contrary case, that is, if $|f_1| \ge |f_2|$, then $|F_1| \le |F_2|$.

Definition 4. If $|f_1 \ominus f_2| = |f_1|$, then $|f_1| = |f_2| = \left| f^* \right|$ and if $b_1 \le b_2$ then $|F_1| > |F_2|$, where $f_1 = \dfrac{a_1}{b_1}$, $f_2 = \dfrac{a_2}{b_2}$. In this case we say that the point f^* is a **multivalent point**.

Definition 5. If $|f_1 \ominus f_2| = |F_1|$, then $|F_1| = |F_2| = \left|F^*\right|$. Then in this case the point F^* is called an **Invariant point**.

4. The Sums Under The Different Strategies

In this section, we will deal with three distinct strategies with respect to the sum and we will compare them, and finally we will examine whether the sums are equal under local and global criteria.

τ_n cortege is defined as

$$\tau_n = \left\{ f_i \middle| \ i \in w_n, w_n = 1, 2, ..., n, |f_i| \ge |f_{i+1}|, \ i = 1, ..., \mathrm{n}-1 \right\}.$$

F_K^L defined by

$$F_K^L = \overset{k}{\underset{i=1}{\oplus}} f_i$$

denotes Local criterium according to the sums where $f_i \in \tau_n$. This means that F_K^L is coordinatewise sum of the first k elements ordered by their module.

$w_k^L = \{1, 2, ..., k\}$ denotes the ordered of first k index set.

Given

$$F_1^G = f_1,$$
$$F_{k+1}^G = \left\{ F_k^G \oplus f_j \middle| \ \left|F_k^G \oplus f_j\right| = \max_{i \in w_n \setminus w_k^G} \left\{ \left|F_k^G \oplus f_i\right| \right\} \right\},$$
$$w_1^G = \{1\},$$
$$w_{k+1}^G = \left\{ w_k^G \cup j \middle| \ F_{k+1}^G = F_k^G \oplus f_j \right\}.$$

F_k^G denotes the sum according to global criterium. F_{k+1}^G inserts automatically F_k^G into F_{k+1}^G and chooses among the remaining elements as (k+1)th element f_j which makes the coordinatewise sum maximum.

F_k^* chooses k f_i fractions which make coordinatewise sum maximum if $k < n$.
w_k^* is taken to be the set if indicates of f_i chosen from F_k^*.
F_k^* is defined as

$$F_k^* = \left| \underset{i \in w_k^*}{\oplus} f_i \right| = \max_{w_k \subset w_n} \left\{ \left| \underset{i \in w_k}{\oplus} f_i \right| \right\}.$$

The sums may not always give maximal values according to Global and Local criteria. We give now a theorem that indicates a condition under which the sums are equal according to local and global criteria.

Theorem 3. If $\left| f_i \ominus f_j \right| \notin U_{f_i f_j}$ for all $i, j \in w_n$, then $w_k^G = w_k^L = w_k^*$ for every k.

Proof. The proof of $w_k^G = w_k^L$ follows from the algorithmic construction of sets w_k^G and w_k^L when conditions of the theorem are considered. We prove $w_k^G = w_k^*$ by induction. Clearly $w_1^G = w_1^*$ holds for $k = 1$.

Assume that theorem holds for $k = m-1$, that is, $w_{m-1}^G = w_{m-1}^*$. Suppose that $w_m^G = w_m^*$ does not hold. Then there exists $\overline{w_m}$ such that $\left| \underset{i \in \overline{w_m}}{\oplus} f_i \right| > \left| \underset{i \in w_m^G}{\oplus} f_i \right|$. Then we have two cases $\overline{w_m} \cap w_m^G = \varnothing$ and $\overline{w_m} \cap w_m^G \neq \varnothing$.

Case 1: Let $\overline{w_m} \cap w_m^G = \varnothing$. Then

$$\left| \left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \oplus f_s \right| \leq \left| \left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \oplus f_p \right| \tag{12}$$

,for some $\exists p \in \overline{w_m}$ and $\forall s \in \overline{w_m}$. By the definition of w_{k+1}^G, we have

$$\left| \left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \oplus f_p \right| \leq \left| \left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \oplus f_m \right| = \left| \underset{i \in w_m^G}{\oplus} f_i \right| = F_m^G \tag{13}$$

Using Proposition 1 and (13), we get

$$\left| \underset{t \in \overline{w_m}}{\oplus} \left[\left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \oplus f_t \right] \right| \leq \left| \left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \oplus f_p \right|$$

$$\left| \left[\underset{t \in \overline{w_m}}{\oplus} \underset{i \in w_{m-1}^G}{\oplus} f_i \right] \oplus \left[\underset{t \in \overline{w_m}}{\oplus} f_t \right] \right| \leq \left| \left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \oplus f_p \right|$$

$$\left| \left[m \otimes \left(\underset{i \in w_{m-1}^G}{\oplus} f_t \right) \right] \oplus \left[\underset{t \in \overline{w_m}}{\oplus} f_t \right] \right| \leq \left| \underset{i \in w_m^G}{\oplus} f_i \right| = F_m^G \tag{14}$$

$$\left| m \otimes \left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \right| = \left| \left(\underset{i \in w_{m-1}^G}{\oplus} f_i \right) \right| = F_{m-1}^G. \tag{15}$$

From (14) and (15) it follows that

$$F_m^G \leq F_{m-1}^G. \tag{16}$$

The following inequality is clear from Proposition 4 and (16)

$$\left| \underset{t \in \overline{w_m}}{\oplus} f_t \right| \leq \left| \underset{i \in w_m^G}{\oplus} f_i \right| \tag{17}$$

But (17) is contradicts assumption (12). Therefore, the theorem is proved for this case.

Case 2. Let $\overline{w_m} \cap w_m^G \neq \varnothing$. For the sake of simplicity, consider only the case $m = 3$. We can write such a proof for all $m < n$.

Let $\overline{w_3} = \{i_1, i_2, i_3\}$, $w_3^G = \{1,2,3\}$, $w_2^G = \{1,2\}$. Without loss of generality, let $i_1 = 1$ i.e., $\overline{w_3} \cap w_2^G = 1$, $i_2 = p$ and

$$\left|\left(\underset{i\in w_2^G}{\oplus} f_i\right) \oplus f_{i_3}\right| \leq \left|\left(\underset{i\in w_2^G}{\oplus} f_i\right) \oplus f_{i_2}\right|$$

By the definition of w_{k+1}^G, we have

$$\left|\left(\underset{i\in w_2^G}{\oplus} f_i\right) \oplus f_{i_2}\right| \leq \left|\left(\underset{i\in w_2^G}{\oplus} f_i\right) \oplus f_3\right| = \left|\underset{i\in w_3^G}{\oplus} f_i\right| = F_3^G.$$

Using associativitiy and commutativity properties of $\oplus$ we get, by developing (15),

$$\left\|\left[\left(f_1 \oplus f_2\right) \oplus f_2\right] \oplus \left[f_1 \oplus \left(f_{i_2} \oplus f_{i_3}\right)\right]\right\| = \left\|\left[\left(f_1 \oplus f_2\right) \oplus f_{i_2}\right] \oplus \left[\left(f_1 \oplus f_2\right) \oplus f_{i_3}\right]\right\|$$

Using Proposition 1 we obtain from inequality above

$$\left\|\left[\left(f_1 \oplus f_2\right) \oplus f_{i_2}\right] \oplus \left[\left(f_1 \oplus f_2\right) \oplus f_{i_3}\right]\right\| \leq \left|\left(f_1 \oplus f_2\right) \oplus f_{i_2}\right| \tag{18}$$

From the condition $\left|f_2 \ominus f_3\right| \notin U_{f_2 f_3}$ in the theorem it follows that we have

$$\left|f_1 \oplus f_{i_2} \oplus f_{i_3}\right| \geq \left|f_1 \oplus f_2 \oplus f_3\right|. \tag{19}$$

Taking in the consideration (18) and (19) we obtain

$\left|f_1 \oplus f_{i_2} \oplus f_{i_3}\right| \leq \left|f_1 \oplus f_2 \oplus f_3\right|$, which is in contradiction with (19).

5. Some Propositions

It is important to know relation between F_k^G defined according to global strategy and F_k^L defined by local strategy. We have already proved $\left|F_2^G\right| > \left|F_2^L\right|$ and $\left|F_3^G\right| < \left|F_3^L\right|$. Throughout this section we set $f_i = \dfrac{a_i}{b_i}$.

Proposition 5. If we define $B_k^G = \sum_{i\in w_k^G} b_i$ and $B_k^L = \sum_{i\in w_k^L} b_i$ for each $k < n$, then $B_k^G \leq B_k^L$.

Proof. We proceed by induction. The result holds for $k = 1$, since

$$B_1^G = \sum_{i\in w_1^G} b_i = \sum_{i\in w_1^L} b_i = B_1^L$$

where $w_1^G = 1 = w_1^L$.

Assume that we have

$$B_m^G = \sum_{t\in w_m^G} b_t = b_{t_1} + b_{t_2} + \ldots + b_{t_m}$$

$$B_m^L = \sum_{t\in w_m^G} b_t = b_1 + b_2 + \ldots + b_m$$

For $k = m$, and prove the proposition for $k = m+1$.
Now according to local and global strategies we have by definition of sum

$$B_{m+1}^L = B_m^L + b_{m+1}$$

and

$$B_{m+1}^G = B_m + b_{i_{m+1}}.$$

There are three cases to consider.

(i) If $i_{m+1} = m+1$, then $B_{m+1}^G \le B_{m+1}^L$. Indeed, by induction hypothesis, it sufficies to add $m+1$ on both sides.

(ii) If $i_{m+1} < m+1$, then there exists $t \in w_m^G$ such that $i_{m+1} = t$. Then $i_t \in w_m^L$. If $i_t < m+1$, then there is $s \in w_m^G$ such that $i_t = s$, and in this case $i_s \in w_m^L$. When $i_s > m+1$, clearly we have it is obvious that $\left| f_{i_t} \ominus f_t \right| \in U_{f_{i_t} f_t}$ and $\left| f_{i_s} \ominus f_s \right| \in U_{f_{i_s} f_s}$. In this case, $b_{i_s} \le b_s$ and we conclude that $B_{m+1}^G \le B_{m+1}^L$.

(iii) If $i_{m+1} > m+1$, then $\left| f_{i_{m+1}} \ominus f_{m+1} \right| \in U_{f_{i_{m+1}} f_{m+1}}$; hence $b_{m+1} \ge b_{i_{m+1}}$, and we get $B_{m+1}^G \le B_{m+1}^L$.

Proposition 6. We have $\left|F_k^G\right| \ge |f_k|$ and $B_k^G > b_k$, for every $k \le n$.

Proof. The part $B_k^G > b_k$ is obvious. For the proof of $\left|F_k^G\right| \ge |f_k|$, we employ induction ok k. For $k = 1$, according to global strategy by definition of sum we have $F_1^G = f_1$, hence $\left|F_1^G\right| = |f_1|$.

Assume that $\left|F_m^G\right| \ge |f_m|$ for $k = m$ and prove that $\left|F_{m+1}^G\right| \ge |f_{m+1}|$ for $k = m+1$.

There is a f_s such that $F_{m+1}^G = F_m \oplus f_s$.

(a) If $s = m+1$, then by Proposition 1, we have $\left|F_{m+1}^G\right| \ge |f_{m+1}|$.

(b) If $s \ne m+1$, then $m+1 \in w_m^*$. Assume, to the contrary that

$$\left|F_m^G \oplus f_s\right| < |f_{m+1}| \tag{20}$$

Now by induction, we have

$$\left|F_m^G\right| \ge |f_m| \ge |f_{m+1}| \tag{21}$$

and

$$\left|F_m^G \oplus f_{m+1}\right| \ge |f_{m+1}|. \tag{22}$$

Therefore, we get $\left|F_m^G \oplus f_{m+1}\right| > \left|f_m^G \oplus f_s\right|$ from (21) and (22). But this contradicts (20), thus we obtain the inequality $\left|F_m^G \oplus f_s\right| \ge |f_{m+1}|$.

Proposition 7. We have $\left|F_k^G\right| \geq \left|F_{k+1}^G\right|$ and $B_k^G < B_{k+1}^G$, for every $k < n$; that is, $\left\{F_k^G\right\}_{k=1}^n$ is monotone decreasing while $\left\{B_k^G\right\}_{k=1}^n$ is monotone increasing.

Proof. Clearly $B_k^G < B_{k+1}^G$ holds. We prove the first part by induction on k.

For k=1 there is f_s such that $F_1^G = f_1,\ F_2^G = f_1 \oplus f_s$. Since $\left|f_1\right| \geq \left|f_1 \oplus f_s\right|$ by Proposition 1, we then have $F_1^G \geq F_2$. Assume that $\left|F_m^G\right| \geq \left|F_{m+1}^G\right|$ holds for k=m and prove $\left|F_{m+1}^G\right| \geq \left|F_{m+2}^G\right|$.

In other words, if we show that $\left|F_m^G\right| \geq \left|f_s\right|$, then $\left|F_m^G\right| \geq \left|F_m^G \oplus f_s\right|$ will follow from Proposition 1. This means that we thus get $\left|F_m^G\right| \geq \left|F_{m+1}^G\right|$. To prove $\left|F_m^G\right| \geq \left|f_s\right|$ we consider the following.

(i) Let $s \geq m$. Then by Proposition 6 we find $\left|F_m^G\right| \geq \left|f_m\right| \geq \left|f_s\right|$.

(ii) For $s < m$ we have $\left|F_m^G\right| < \left|f_s\right|$. Define $\overline{F_m}$ by

$$\overline{F_m} = F_{m-1}^G \oplus f_s \tag{23}$$

From our assumption we have

$$F_{m-1}^G \geq F_m^G. \tag{24}$$

Now

$$\left|F_{m-1}^G \oplus f_s\right| \geq \left|F_m^G\right|$$

Follows from (23), (24) and Proposition 1. This is $\overline{F_m} \geq F_m^G$. But it contradicts with F_m^G to be maximum. Therefore, we find $\left|F_m^G\right| \geq \left|f_s\right|$.

Proposition 8. If $\max\limits_{i \in \underline{w_k}}\{i\} < \min\limits_{i \in w_k}\{i\}$ for all $\overline{w_k}, w_k \in w_n$ then $\left|\bigoplus\limits_{i \in \underline{w_k}} f_i\right| \geq \left|\bigoplus\limits_{i \in w_k} f_i\right|$.

Proof. Let $\max\limits_{i \in \underline{w_k}}\{i\} < i_1$ and $\min\limits_{i \in w_k}\{i\} = i_2$. By Proposition 6 we get

$$\left|\bigoplus_{i \in \underline{w_k}} f_i\right| \geq f_{i_1}. \tag{25}$$

From Proposition 6 again, we find

$$\left|\bigoplus_{i \in w_k} f_i\right| \geq f_{i_2}. \tag{26}$$

Then we see by the hypothesis that $\max\limits_{i \in \underline{w_k}}\{i\} < \min\limits_{i \in w_k}\{i\}$, $\left|f_{i_1}\right| \geq \left|f_{i_2}\right|$. The desired inequality follows from (25) and (26).

6. Conclusion

In this work, we prove that, in fractions algebra formed by coordinatewise operations defined on fractions, the sum operation possesses inversion property in some cases. The inequality condition of sums is investigated according to different strategies. Then taking into account this condition, we prove some inequalities which allow to make use the difference of sums compared by different strategies from the maximal sum. These inequalities will be use to evaluate solutions of LFP found by heuristic algorithms based on suggested strategies above.

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